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T_∞-Fuzzy Observables

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Observables are defined as homomorphisms from the Borel σ-algebra into a family of fuzzy sets considered with respect to the Giles connectives. Algebraic operations with observables are introduced and their relation to the corresponding operations with fuzzy random variables is explained.

1. INTRODUCTION

In Kolesárová and Riečan (1992) we introduced for any measurable t-norm T on $\langle 0, 1\rangle^2$ a T-fuzzy observable as a mapping x from $\mathscr{B}(\mathbb{R})$ into a generated fuzzy σ-algebra τ of fuzzy subsets of a given universum Ω, satisfying the following properties: $x(E^c) = x(E)' = 1 - x(E)$ for each $E \in B(\mathbb{R})$ and $x(\bigcup_{n\in N} E_n) = S_{n\in N}(x(E_n))$ for each sequence $\{E_n\}_{n\in N} \subset \mathscr{B}(\mathbb{R})$, $E_i \cap E_j = \emptyset$ for $i \neq j$ (S denotes a dual t-conorm of T).

We have shown that if T is an Archimedean t-norm, then x also preserves a maximal and a minimal element (Kolesárová and Riečan, 1992, Proposition 1), i.e., x is a homomorphism from $\mathscr{B}(\mathbb{R})$ into τ. If T is a strict t-norm, then any T-fuzzy observable x is an inverse of a crisp random variable (Kolesárová and Riečan, 1992, Proposition 2). So, the most interesting are T-fuzzy observables which are induced by Archimedean nonstrict t-norms. Since each Archimedean nonstrict t-norm T can be obtained by a transformation of the fundamental t-norm T_∞, $T_\infty(x, y) = \max(x + y - 1, 0)$, we will pay attention only to the T_∞-fuzzy observables [for more details about t-norms see Schweizer and Sklar, (1983) or Kolesárová and Riečan, (1992)].

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The t-norm T_∞ induces the Giles bold intersection of fuzzy sets: $u \Cap v = \max(u+v-1, 0)$ corresponding to the Lukasiewicz conjunction. (Recall that a fuzzy subset u of a universum Ω is a mapping $u\colon \Omega \to \langle 0, 1\rangle$). The dual t-norm S_∞, $S_\infty(x, y) = \min(x+y, 1)$ induces the Giles bold union of fuzzy sets: $u \Cup v = \min(u+v, 1)$. Throughout this paper just these fuzzy connectives will be used. Note that mentioned fuzzy connectives were proposed by Pykacz (1991) for fuzzy modeling of quantum mechanics.

2. T_∞-FUZZY OBSERVABLES AND THEIR CALCULUS

Let $(\Omega, \mathscr{S})$ be a measurable space, i.e., let Ω be an arbitrary nonempty set and let $\mathscr{S}$ be a σ-algebra of its crisp subsets. Let $\tau \subset \langle 0, 1\rangle^\Omega$ be a generated fuzzy σ-algebra, i.e., the system of all $\mathscr{S} - \mathscr{B}(\langle 0, 1\rangle)$ measurable fuzzy subsets of Ω.

Definition 1. A mapping $x\colon \mathscr{B}(\mathbb{R}) \to \tau$ is said to be a T_∞-fuzzy observable if:

(i) $x(E^c) = x(E)' = 1 - x(E)$ for each $E \in \mathscr{B}(\mathbb{R})$.

(ii)

$$x\left(\bigcup_{n\in N} E_n\right) = \Cup_{n\in N}\, x(E_n) = \min\left(\sum_{n\in N} x(E_n), 1\right)$$

for each sequence $\{E_n\}_{n\in N} \subset \mathscr{B}(\mathbb{R})$, $E_i \cap E_j = \emptyset$ for $i \neq j$. [$\mathscr{B}(\mathbb{R})$ is the system of all Borel subsets of the real line.]

Since T_∞ is an Archimedean t-norm, T_∞-fuzzy observables have the following property (Kolesárová and Riečan, 1992, Proposition 1).

Lemma 1. Let x be a T_∞-fuzzy observable. Then $x(\mathbb{R}) = 1_\Omega$ and $x(\emptyset) = 0_\Omega$.

This means that a T_∞-fuzzy observable is a σ-homomorphism. Moreover, property (ii) in Definition 1 can be expressed in the following form:

Lemma 2. Let x be a T_∞-fuzzy observable. Then for each sequence $\{E_n\}_{n\in N} \subset \mathscr{B}(\mathbb{R})$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, it holds that

$$x\left(\bigcup_{n\in N} E_n\right) = \sum_{n\in N} x(E_n).$$

Proof. See the proof of Proposition 3 in Kolesárová and Riečan (1992, (iii)). ■

Each T_∞-fuzzy observable x induces a system $\mathcal{U} = \{x((-\infty, t));\, t \in \mathbb{R}\}$ of fuzzy subsets which has these properties:

(P1) $x((-\infty, t)) = \sup_{r \in Q, r < t} x((-\infty, r))$, where Q is the set of all rational numbers.

(P2) $\inf_{r \in Q} x((-\infty, r)) = 0$.

(P3) $\sup_{r \in Q} x((-\infty, r)) = 1$.

Conversely, each system $\mathcal{U} = \{u_t;\, t \in \mathbb{R}\}$ of fuzzy subsets of τ fulfilling the properties (P1)–(P3) determines a T_∞-fuzzy observable x given by

$$x((-\infty, t)) = u_t, \qquad t \in \mathbb{R}$$

Note that τ is a closed system under countable infima and suprema (Butnariu and Klement, 1991).

Let x and y be T_∞-fuzzy observables. Let us put

$$z_t = \bigvee_{r \in Q} [x((-\infty, r)) \wedge y((-\infty, t - r))] \tag{1}$$

Lemma 3. The system $\mathcal{Z} = \{z_t;\, t \in \mathbb{R}\} \subset \tau$ fulfills the properties (P1)–(P3).

Proof. (i) The property $z_t = \sup_{r \in Q, r < t} z_r$ follows immediately from the property (P1) of T_∞-fuzzy observables x and y and equation (1).

(ii) Let ω be an arbitrary but fixed element of Ω. Since x is a T_∞-fuzzy observable, it fulfills the property (P2) and therefore for each $\varepsilon > 0$ there exists $q_1 \in Q$ such that

$$0 \leq x((-\infty, q_1))(\omega) < \frac{\varepsilon}{2}$$

As x is monotone,

$$x((-\infty, q))(\omega) < \frac{\varepsilon}{2} \qquad \text{holds for each} \quad q \leq q_1$$

Analogously for a T_∞-fuzzy observable y we get

$$y((-\infty, q))(\omega) < \frac{\varepsilon}{2} \qquad \text{for each} \quad q \leq q_2$$

Let us put $q_0 = \min(q_1, q_2)$. Let $r \in Q$. Then either $r \leq q_0$ or $q_0 < r$. If $r \leq q_0$, then

$$x((-\infty, r))(\omega) < \frac{\varepsilon}{2}$$

If $q_0 < r$, then $2q_0 - r < q_0$ and therefore

$$y((-\infty, 2q_0 - r))(\omega) < \frac{\varepsilon}{2}$$

This means that

$$x((-\infty, r))(\omega) \wedge y((-\infty, 2q_0 - r))(\omega) < \frac{\varepsilon}{2}$$

for each $r \in Q$. Therefore

$$z_{2q_0}(\omega) = \bigvee_{r \in Q} [x((-\infty, r))(\omega) \wedge y((-\infty, 2q_0 - r))(\omega) \leq \frac{\varepsilon}{2} < \varepsilon$$

We have just shown that for each $\varepsilon > 0$ there exists $\bar{q} = 2q_0 \in Q$ such that

$$0 \leq z_{\bar{q}}(\omega) < \varepsilon$$

and so $\inf_{q \in Q} z_q = 0$ and the property (P2) is true. The property (P3) can be proved analogously. ■

Since the system $\mathscr{Z}$ fulfills the properties (P1)–(P3), according to the previous part it determines uniquely a T_∞-fuzzy observable z given by

$$z((-\infty, t)) = z_t = \bigvee_{r \in Q} [x((-\infty, r)) \wedge y((-\infty, t - r))] \tag{2}$$

for each $t \in \mathbb{R}$.

Definition 2. A T_∞-fuzzy observable z defined by (2) is called a sum of T_∞-fuzzy observables x and y: $z = x + y$.

Remark 1. A similar approach was used by Dvurečenskij and Tirpáková (1988) for introducing a sum of two T_0-fuzzy observables. Note that in this case the Zadeh fuzzy connectives were used. T_0-fuzzy observables are not complete homomorphisms, up to the crisp case.

In the next part we will show how it is possible to introduce other operations for T_∞-fuzzy observables.

Let $h: \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function and let $x: \mathscr{B}(\mathbb{R}) \to \tau$ be a T_∞-fuzzy observable. Then a mapping $hx: \mathscr{B}(\mathbb{R}) \to \tau$ defined by

$$hx(E) = x(h^{-1}(E)), \qquad E \in \mathscr{B}(\mathbb{R})$$

is again a T_∞-fuzzy observable. For example,

$$x^2(E) = x(\{t \in \mathbb{R};\, t^2 \in E\})$$

$$cx(E) = x(\{t \in \mathbb{R};\, ct \in E\}), \qquad c \in \mathbb{R} \tag{3}$$

In particular

$$-x(E) = x(\{t \in \mathbb{R}:\, -t \in E\}) = x(-E) \tag{4}$$

That is why we are able to introduce a difference and a product of two

T_∞-fuzzy observables $\boldsymbol{x}$ and $\boldsymbol{y}$, following the ideas of von Neumann for observables and Dvurečenskij for T_0-fuzzy observables, in this way:

$$\boldsymbol{x}-\boldsymbol{y}=\boldsymbol{x}+(-\boldsymbol{y}) \tag{5}$$

$$\boldsymbol{x}\cdot\boldsymbol{y}=\tfrac{1}{2}[(\boldsymbol{x}+\boldsymbol{y})^2-\boldsymbol{x}^2-\boldsymbol{y}^2] \tag{6}$$

Definition 3. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be T_∞-fuzzy observables. An observable $\boldsymbol{x}$ is dominated by an observable $\boldsymbol{y}$: $\boldsymbol{x}\prec\boldsymbol{y}$, if

$$\boldsymbol{x}((-\infty,t))\geqq\boldsymbol{y}((-\infty,t)) \qquad \text{for each} \quad t\in\mathbb{R}$$

Let us note that if u, v are two fuzzy sets, then $u\le v \Leftrightarrow u(\omega)\le v(\omega)$ for each $\omega\in\Omega$.

A T_∞-fuzzy observable $\boldsymbol{x}_0$ will be called the *zero observable* if $\boldsymbol{x}_0(\{0\})=1_\Omega$. In other words, if

$$\boldsymbol{x}_0(E)=\begin{cases}0_\Omega & 0\notin E\\ 1_\Omega & 0\in E\end{cases}$$

for each $E\in\mathscr{B}(\mathbb{R})$.

Evidently $\boldsymbol{x}_0=-\boldsymbol{x}_0=\boldsymbol{x}_0^2$. Note that $\boldsymbol{x}=-\boldsymbol{x}$ does not imply $\boldsymbol{x}=\boldsymbol{x}_0$. Further, since

$$\boldsymbol{x}_0((-\infty,t))=\begin{cases}0_\Omega & t\le 0\\ 1_\Omega & t>0\end{cases}$$

$$\boldsymbol{x}^2((-\infty,t))=\begin{cases}0_\Omega & t\le 0\\ \boldsymbol{x}((-\sqrt{t},\sqrt{t}))\le 1_\Omega & t>0\end{cases}$$

then for each T_∞-fuzzy observable $\boldsymbol{x}$ it holds that $\boldsymbol{x}_0\prec\boldsymbol{x}^2$.

If $\boldsymbol{x}_0\prec\boldsymbol{x}$ we shall also use the expression: an observable $\boldsymbol{x}$ is nonnegative.

Finally, the sum of a T_∞-fuzzy observable $\boldsymbol{x}$ and the zero observable $\boldsymbol{x}_0$ is given by

$$(\boldsymbol{x}_0+\boldsymbol{x})((-\infty,t))=\bigvee_{r\in Q}[\boldsymbol{x}_0((-\infty,r))\wedge\boldsymbol{x}((-\infty,t-r))]$$

If $r\le 0$ then $\boldsymbol{x}_0((-\infty,r))=0_\Omega$ and so it is enough to deal with $r\in Q$, $r>0$. So, let $r>0$. Then $\boldsymbol{x}_0((-\infty,r))=1_\Omega$ and $\boldsymbol{x}_0((-\infty,r))\wedge\boldsymbol{x}((-\infty,t-r))=\boldsymbol{x}((-\infty,t-r))$.

Therefore

$$(\boldsymbol{x}_0+\boldsymbol{x})((-\infty,t))=\bigvee_{\substack{r\in Q\\ r>0}}\boldsymbol{x}((-\infty,t-r))=\boldsymbol{x}((-\infty,t))$$

for each $t\in\mathbb{R}$ and this means that the equality $\boldsymbol{x}_0+\boldsymbol{x}=\boldsymbol{x}$ holds for each T_∞-fuzzy observable $\boldsymbol{x}$.

3. FUZZY-VALUED RANDOM VARIABLES

Following the ideas of Höhle (1976, 1981), Rodabaugh (1982), and others, Klement (1985, 1987) introduced the concept of fuzzy-valued functions. We recall some basic notions. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $I = \langle 0, 1 \rangle$. The extended fuzzy real line $\bar{\mathbb{R}}(I)$ is the set of all functions $p: \bar{\mathbb{R}} \to I$ such that:

(i) $p(-\infty) = 0$ and $p(+\infty) = 1$.
(ii) $p(r) = \sup\{p(s); s < r, s \in \mathbb{R}\}$ for each $r \in \mathbb{R}$.

Note that a fuzzy real number $p \in \bar{\mathbb{R}}(I)$ is a cumulative distribution function on $\bar{\mathbb{R}}$. A fuzzy number p can be interpreted as follows: $p(r)$ is a degree at which p is less than (nonfuzzy) number r. A nonfuzzy number r is identified with the characteristic function of the set $(r, \infty\rangle$. A fuzzy number p is said to be finite if $\inf\{p(r); r \in \mathbb{R}\} = 0$ and $\sup\{p(r); r \in \mathbb{R}\} = 1$. A finite fuzzy number is a cumulative distribution on $\mathbb{R}$ and vice versa. The set of all finite fuzzy numbers will be denoted by $\mathbb{R}(I)$.

The partial ordering $\angle$ on $\bar{\mathbb{R}}(I)$ is given by

$$p \angle u \Leftrightarrow \forall r \in \bar{\mathbb{R}}: \quad p(r) \geq u(r) \tag{7}$$

Now, let $f: \langle a, b \rangle \to \langle c, d \rangle$ be a nondecreasing function, left-continuous in (a, b) with $f(a) = c$. Then the quasi-inverse of f is a function $[f]^q: \langle c, d \rangle \to \langle a, b \rangle$ defined by

$$[f]^q(s) = \sup\{r \in \langle a, b \rangle; f(r) < s\}, \qquad \text{for} \quad s \in (c, d\rangle$$

$$[f]^q(c) = a$$

The quasi-inverse of f is again a nondecreasing function, left-continuous in (c, d) and $[[f]^q]^q = f$. The set of all quasi-inverses of fuzzy numbers $p \in \bar{\mathbb{R}}(I)$ will be denoted by $\bar{\mathbb{R}}^q(I)$.

Due to the fact that the mapping $q: p \mapsto [p]^q$ is an involution from $\bar{\mathbb{R}}(I)$ onto $\bar{\mathbb{R}}^q(I)$, it is possible to introduce an algebraic structure on $\bar{\mathbb{R}}(I)$ as follows:

Let $p, u \in \bar{\mathbb{R}}(I)$. Then

$$p \angle u \Leftrightarrow [p]^q(\alpha) \leq [u]^q(\alpha) \qquad \text{for all} \quad \alpha \in I \tag{8}$$

$$[p \oplus u]^q(\alpha) = [p]^q(\alpha) + [u]^q(\alpha) \tag{9}$$

$$[p \otimes u]^q(\alpha) = \sup\{[p^+]^q(\beta) \cdot [u^+]^q(\beta) + [p^+]^q(1-\beta) \cdot [u^-]^q(\beta) + [p^-](\beta) \cdot [u^+]^q(1-\beta) + [p^-]^q(1-\beta) \cdot [u^-]^q(1-\beta); \; \beta < \alpha\} \tag{10}$$

where

$$p^+(r) = \begin{cases} 0, & r \leq 0 \\ p(r), & r > 0 \end{cases}$$

$$p^-(r) = \begin{cases} p(r), & r \leq 0 \\ 1, & r > 0 \end{cases}$$

The previous formulas for $p \oplus u$ and $p \otimes u$ can be used if their right-hand sides make sense.

$\bar{\mathbb{R}}(I)$ can be considered as a subspace of $\langle 0, 1\rangle^{\bar{\mathbb{R}}}$. Thus we can equip it with the product σ-algebra and it makes sense to consider measurable functions $X: \Omega \to \bar{\mathbb{R}}(I)$, which we will call fuzzy-valued random variables (measurability of these functions is defined as usual).

By Proposition 2.1 in Klement (1975), the measurability of a function $X: \Omega \to \bar{\mathbb{R}}(I)$ is equivalent to the existence of a Markov kernel $\mathscr{K}$ from $(\Omega, \mathscr{S})$ to $(\bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ such that for all $(\omega, t) \in \Omega \times \bar{\mathbb{R}}$, $X(\omega)(t) = \mathscr{K}(\omega, \langle -\infty, t))$.

Note that Klement (1975) deals only with nonnegative fuzzy numbers. The extension to $\bar{\mathbb{R}}(I)$ is evident.

4. FINITE FUZZY-VALUED RANDOM VARIABLES AND T_∞-FUZZY OBSERVABLES

There exists a one-to-one correspondence between finite fuzzy-valued random variables [i.e., with values in $\mathbb{R}(I)$] and T_∞-fuzzy observables [proved in Kolesárová and Riečan (1992)]. The correspondence between a T_∞-fuzzy observable $\boldsymbol{x}: \mathscr{B}(\mathbb{R}) \to \tau$ and a fuzzy-valued random variable $X: \Omega \to \mathbb{R}(I)$ is expressed by the formula

$$X(\omega)(t) = \boldsymbol{x}((-\infty, t)(\omega) \tag{11}$$

for each $t \in \mathbb{R}$ and $\omega \in \Omega$.

The reciprocal correspondence between $\boldsymbol{x}$ and X will be denoted by $\boldsymbol{x} \leftrightarrow X$.

Now, let X_a, $a \in \mathbb{R}$, be a fuzzy random variable defined by

$$X_a(\omega) = \mathbf{1}_{(a,\infty)}$$

for each $\omega \in \Omega$.

Due to (11), a fuzzy random variable X_a corresponds to a T_∞-fuzzy observable $\boldsymbol{x}_a$ which is given by $\boldsymbol{x}_a(\{a\}) = 1_\Omega$.

In particular, for $a = 0$ we get the zero observable $\boldsymbol{x}_0$.

Let $\boldsymbol{x} \leftrightarrow X, \boldsymbol{y} \leftrightarrow Y$. Let $\boldsymbol{x} + \boldsymbol{y}$ be the sum of T_∞-fuzzy observables $\boldsymbol{x}$ and $\boldsymbol{y}$ created by (2) and let $X + Y$ be the sum of fuzzy random variables. Its value $(X + Y)(\omega) = X(\omega) \oplus Y(\omega)$ is defined by (9).

Theorem 1. Let $\boldsymbol{x} \leftrightarrow X$ and $\boldsymbol{y} \leftrightarrow Y$. Then $\boldsymbol{x} + \boldsymbol{y} \leftrightarrow X + Y$.

Proof. It is necessary to prove that for each $\omega \in \Omega$ and $t \in \mathbb{R}$,

$$(X + Y)(\omega)(t) = (\boldsymbol{x} + \boldsymbol{y})((-\infty, t))(\omega)$$

For simplicity let us denote $X(\omega) = p$, $Y(\omega) = s$, $(p \oplus s)(t) = \gamma$, and $(\boldsymbol{x} + \boldsymbol{y})((-\infty, t))(\omega) = \beta$. This means we have to prove $\beta = \gamma$.

(i) According to (2), we have

$$\beta = \bigvee_{r \in Q} [\boldsymbol{x}((-\infty, r))(\omega) \wedge \boldsymbol{y}((-\infty, t - r))(\omega)]$$

From the properties of the supremum we get that for each $\varepsilon > 0$ there exists $r_\varepsilon \in Q$ such that

$$\boldsymbol{x}((-\infty, r_\varepsilon))(\omega) \wedge \boldsymbol{y}((-\infty, t - r_\varepsilon))(\omega) > \beta - \varepsilon$$

If we take into account the assumption $\boldsymbol{x} \leftrightarrow X$, $\boldsymbol{y} \leftrightarrow Y$ and the introduced designation we get

$$p(r_\varepsilon) \wedge s(t - r_\varepsilon) > \beta - \varepsilon \tag{12}$$

Further,

$$\begin{aligned} \gamma = (p \oplus s)(t) &= [[p \oplus s]^q]^q(t) \\ &= \sup\{\alpha; [p \oplus s]^q(\alpha) < t\} \\ &= \sup\{\alpha; \sup\{u; p(u) < \alpha\} + \sup\{v; s(v) < \alpha\} < t\} \end{aligned} \tag{13}$$

If we take into account (12), we get

$$\sup\{u; p(u) < \beta - \varepsilon\} < r_\varepsilon$$

$$\sup\{v; s(v) < \beta - \varepsilon\} < t - r_\varepsilon$$

and so $\beta - \varepsilon \in \{\alpha; [p]^q(\alpha) + [s]^q(\alpha) < t\}$.

Therefore $\beta - \varepsilon \le \gamma$. Since the last inequality holds for each $\varepsilon > 0$, we have

$$\beta \le \gamma \tag{14}$$

(ii) For each $\varepsilon > 0$ there exists $\alpha_\varepsilon > \gamma - \varepsilon$ such that

$$\sup\{u; p(u) < \alpha_\varepsilon\} + \sup\{v; s(v) < \alpha_\varepsilon\} < t$$

[this fact follows from (13)].

Let us put $\sup\{u; p(u) < \alpha_\varepsilon\} = u_0$ and $\sup\{v; s(v) < \alpha_\varepsilon\} = v_0$. Then we can write $u_0 + v_0 < t$, $p(u_0) \ge \alpha_\varepsilon$, $s(v_0) \ge \alpha_\varepsilon$. Moreover, for each $\delta > 0$,

$$p(u_0 + \delta) \ge \alpha_\varepsilon \qquad \text{and} \qquad s(v_0 + \delta) \ge \alpha_\varepsilon$$

We can choose such $r \in Q$ that

$$u_0 < r \qquad \text{and} \qquad v_0 < t - r$$

For this value we have

$$p(r) \wedge s(t-r) \geq \alpha_\varepsilon$$

and therefore $\beta \geq \alpha_\varepsilon$.

Since $\alpha_\varepsilon > \gamma - \varepsilon$, we get the inequality $\beta > \gamma - \varepsilon$, which holds for each $\varepsilon > 0$. Therefore $\beta \geq \gamma$.

The last result together with (14) mean that the assertion of Theorem 1 is true. ■

Let us notice the ordering of fuzzy-valued random variables, in connection with ordering of T_∞-fuzzy observables. It holds that

$$X \leq Y \Leftrightarrow X(\omega) \angle Y(\omega) \qquad \text{for each} \quad \omega \in \Omega$$

By (7), $X(\omega) \angle Y(\omega) \Leftrightarrow X(\omega)(t) \geq Y(\omega)(t)$ for each $t \in \mathbb{R}$, and this is the same as

$$\boldsymbol{x}((-\infty, t))(\omega) \geq \boldsymbol{y}((-\infty, t))(\omega)$$

This means that the observable $\boldsymbol{x}$ is dominated by the observable $\boldsymbol{y}$. So

$$X \leq Y \Leftrightarrow \boldsymbol{x} \prec \boldsymbol{y} \tag{15}$$

In contrast with the sum and ordering, in general the product of fuzzy random variables $X \cdot Y$ does not correspond to the product $\boldsymbol{x} \cdot \boldsymbol{y}$ of T_∞-fuzzy observables $\boldsymbol{x}, \boldsymbol{y}$ introduced by (6) (for $\boldsymbol{x} \leftrightarrow X, \boldsymbol{y} \leftrightarrow Y$). Some other facts show that it is not suitable to use the product of T_∞-fuzzy observables defined by (6).

Let $\boldsymbol{x}$ be an arbitrary nonnegative and noncrisp T_∞-fuzzy observable (this means that $\boldsymbol{x}_0 \prec \boldsymbol{x}$ and there exists a set $E \in \mathscr{B}(\mathbb{R})$ for which $\boldsymbol{x}(E)$ is not a crisp subset of Ω). It can be shown that for such T_∞-fuzzy observables the equality $\boldsymbol{x}^2 = \boldsymbol{x} \cdot \boldsymbol{x}$ does not hold. Note that $\boldsymbol{x}^2$ is a T_∞-fuzzy observable created by (3) and $\boldsymbol{x} \cdot \boldsymbol{x}$ by the formula (6).

Example 1. Let $\Omega = \{\omega\}$. Let $\boldsymbol{x}$ be an observable for which

$$\boldsymbol{x}((-\infty, t))(\omega) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ \frac{1}{2} & \text{if} \quad t \in (1, 3\rangle \\ 1 & \text{if} \quad t > 3 \end{cases}$$

Then by (3)

$$\boldsymbol{x}^2((-\infty, t))(\omega) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ \frac{1}{2} & \text{if} \quad t \in (1, 9\rangle \\ 1 & \text{if} \quad t > 9 \end{cases}$$

Using the formula (6), we obtain

$$x \cdot x((-\infty, t))(\omega) = \begin{cases} 0 & \text{if } \ t \le -7 \\ \frac{1}{2} & \text{if } \ t \in (-7, 17\rangle \\ 1 & \text{if } \ t > 17 \end{cases}$$

These results show that for the chosen observable x the equality $x \cdot x = x^2$ is not true.

Proposition 1. Let x be a nonnegative T_∞-fuzzy observable and let X be a finite fuzzy random variable corresponding to x. Then $X \cdot X \leftrightarrow x^2$.

Proof. Let $\omega \in \Omega$ be an arbitrary, but fixed element. Since $x_0 \prec x$ it holds that $x((-\infty, t))(\omega) = 0$ for each $t \le 0$. Therefore

$$x^2((-\infty, t))(\omega) = \begin{cases} 0 & \text{if } t \le 0 \\ x((-\sqrt{t}, \sqrt{t}))(\omega) = x((-\infty, \sqrt{t}))(\omega) & \text{if } t > 0 \end{cases} \tag{16}$$

Let us denote $X \cdot X = X^2$. From the assumptions $x_0 \prec x$ and $X \leftrightarrow x$ we have $X_0 \le X$ and therefore $X^+(\omega) = X(\omega)$ and $X^-(\omega) = \mathbf{1}_{(0,\infty)}$. Then according to (10) it holds that

$$[X^2(\omega)]^q(\alpha) = [X(\omega)]^q(\alpha) \cdot [X(\omega)]^q(\alpha) = ([X(\omega)]^q(\alpha))^2$$

Using this property, we obtain

$$X^2(\omega)(t) = [[X^2(\omega)]^q]^q(t) = \sup\{\alpha; [X^2(\omega)]^q(\alpha) < t\}$$
$$= \sup\{\alpha; ([X(\omega)]^q(\alpha))^2 < t\}$$

Evidently $X^2(\omega)(t) = 0$ for $t \le 0$. If $t > 0$, then

$$X^2(\omega)(t) = \sup\{\alpha; [X(\omega)]^q(\alpha) < \sqrt{t}\} = X(\omega)(\sqrt{t})$$

This means that

$$X^2(\omega)(t) = \begin{cases} 0 & \text{if } \ t \le 0 \\ X(\omega)(\sqrt{t}) & \text{if } \ t > 0 \end{cases}$$

Due to (11) from this result and (16) we obtain $X^2 = X \cdot X \leftrightarrow x^2$. ■

Remark 2. We have shown that for the observable x in Example 1, $x^2 \ne x \cdot x$. Since x is nonnegative, by Proposition 1, $x^2 \leftrightarrow X \cdot X$. Therefore $x \cdot x \not\leftrightarrow X \cdot X$. This property can be proved in general for each nonnegative, noncrisp T_∞-fuzzy observable x.

All these results lead us to the conviction that although it makes sense to define a product of T_∞-fuzzy observables by the formula (6), it is necessary to introduce this operation in another way—because of the not good properties of the mentioned product $x \cdot y$ [given by (6)].

Definition 4. Let $\boldsymbol{x}, \boldsymbol{y}$ be nonnegative T_∞-fuzzy observables. The T_∞-fuzzy observable $\boldsymbol{z}$ defined by

$$\boldsymbol{z}((-\infty, t)) = \begin{cases} 0_\Omega & \text{if } t \le 0 \\ \bigvee_{r \in Q^+} [\boldsymbol{x}((-\infty, r)) \wedge \boldsymbol{y}((-\infty, t/r))] & \text{if } t > 0 \end{cases} \tag{17}$$

will be called the product of $\boldsymbol{x}$ and $\boldsymbol{y}$: $\boldsymbol{z} = \boldsymbol{x} * \boldsymbol{y}$.

We have to prove that Definition 4 is correct. For this purpose it is enough to show that the system $\mathscr{Z} = \{\boldsymbol{z}((-\infty, t)); t \in \mathbb{R}\}$ fulfills the properties (P1)–(P3).

Lemma 4. Let $\boldsymbol{x}, \boldsymbol{y}$ be nonnegative T_∞-fuzzy observables and let $\boldsymbol{x} \leftrightarrow X, \boldsymbol{y} \leftrightarrow Y$. Then

$$(\boldsymbol{x} * \boldsymbol{y})((-\infty, t))(\omega) = (X \cdot Y)(\omega)(t) \tag{18}$$

for each $t \in \mathbb{R}$, $\omega \in \Omega$.

Proof. We omit the details because the assertion can be proved in the same way as Theorem 1. It is enough to replace the sums by products and to write t/r instead of $(t - r)$.

Since $X \cdot Y$ is a finite fuzzy random variable, it corresponds uniquely to a T_∞-fuzzy observable $\boldsymbol{v}$. The correspondence is expressed by (11), i.e.,

$$\boldsymbol{v}((-\infty, t))(\omega) = (X \cdot Y)(\omega)(t)$$

for each $t \in \mathbb{R}$ and $\omega \in \Omega$.

The system $\mathscr{V} = \{\boldsymbol{v}((-\infty, t)); t \in \mathbb{R}\}$ of fuzzy subsets fulfills the properties (P1)–(P3). If we take into account (18), we obtain that also the system $\mathscr{Z} = \{\boldsymbol{x} * \boldsymbol{y}((-\infty, t)); t \in \mathbb{R}\}$ of fuzzy subsets fulfills (P1)–(P3) (because $\mathscr{Z} = \mathscr{V}$). So, this system determines uniquely a T_∞-fuzzy observable $\boldsymbol{z}$ and this fact implies that Definition 4 is correct. Moreover, $\boldsymbol{z} = \boldsymbol{v}$ and therefore the following assertion is true.

Theorem 2. Let $\boldsymbol{x}, \boldsymbol{y}$ be nonnegative T_∞-fuzzy observables and let $\boldsymbol{x} \leftrightarrow X, \boldsymbol{y} \leftrightarrow Y$. Then $\boldsymbol{x} * \boldsymbol{y} \leftrightarrow X \cdot Y$.

Corollary 1. If $\boldsymbol{x}$ is a nonnegative T_∞-fuzzy observable, then $\boldsymbol{x}^2 \leftrightarrow \boldsymbol{x} * \boldsymbol{x}$.

Proof. Let $\boldsymbol{x}_0 \prec \boldsymbol{x}$ and $\boldsymbol{x} \leftrightarrow X$. By Theorem 2, $X \cdot X \leftrightarrow \boldsymbol{x} * \boldsymbol{x}$. According to Proposition 1, the fuzzy observable $\boldsymbol{x}^2$ created by (3) corresponds to fuzzy random variable $X \cdot X$. From the uniqueness of correspondence we obtain

$$\boldsymbol{x} * \boldsymbol{x} = \boldsymbol{x}^2$$

In the previous part of this section fuzzy random variables X_a, $a \in \mathbb{R}$, and T_∞-fuzzy observables $\boldsymbol{x}_a$ (corresponding to X_a) were introduced. Recall that

$$X_1: \quad \Omega \to \mathbb{R}(I), \qquad X_1(\omega) = \mathbf{1}_{(1,\infty)} \qquad \text{for each} \quad \omega \in \Omega$$

and

$$\boldsymbol{x}_1: \quad \mathscr{B}(\mathbb{R}) \to \tau, \qquad \boldsymbol{x}_1(E) = \begin{cases} 0_\Omega & \text{if} \quad 1 \notin E \\ 1_\Omega & \text{if} \quad 1 \in E \end{cases}$$

The fuzzy observable $\boldsymbol{x}_1$ will be called the *unit observable.*

Proposition 2. Let $\boldsymbol{x}$ be a nonnegative T_∞-fuzzy observable. Then

$$\boldsymbol{x} * \boldsymbol{x}_1 = \boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{x} * \boldsymbol{x}_0 = \boldsymbol{x}_0$$

Proof. Let $\boldsymbol{x}_0 \prec \boldsymbol{x}$ and $\boldsymbol{x} \leftrightarrow X$. Let $\omega \in \Omega$ be an arbitrary element. Then

$$X_1^-(\omega) = X_0(\omega) = \mathbf{1}_{(0,\infty)} \qquad \text{and} \qquad X_1^+(\omega) = X_1(\omega) = \mathbf{1}_{(1,\infty)}$$

Therefore

$$[X_1^-(\omega)]^q(\alpha) = 0 \qquad \text{and} \qquad [X_1^+(\omega)]^q(\alpha) = 1 \qquad \text{for each} \quad \alpha \in (0, 1\rangle$$

Let $X(\omega) = p$. Then

$$X \cdot X_1(\omega) = X(\omega) \otimes X_1(\omega) = p \otimes \mathbf{1}_{(1,\infty)}$$

Using (10) for multiplication of fuzzy numbers, we obtain

$$[p \otimes \mathbf{1}_{(1,\infty)}]^q(\alpha) = \sup_{\beta < \alpha} \{[p^+]^q(\beta) + [p^-]^q(\beta)\} = [p]^q(\alpha)$$

for each $\alpha \in (0, 1\rangle$. This means that $X \cdot X_1 = X$. This fact together with the result $X \cdot X_1 \leftrightarrow \boldsymbol{x} * \boldsymbol{x}_1$ following from Theorem 2 makes the assertion $\boldsymbol{x} * \boldsymbol{x}_1 = \boldsymbol{x}$ true. The validity of the property $\boldsymbol{x} * \boldsymbol{x}_0$ is evident.

Let us only note that $\boldsymbol{x}_1$ does not play the role of a unit if the product of T_∞-fuzzy observables is given by (6).

Finally, we will propose how it is possible to define a product of two arbitrary T_∞-fuzzy observables $\boldsymbol{x}$ and $\boldsymbol{y}$.

Each T_∞-fuzzy observable $\boldsymbol{x}$ can be uniquely expressed in the form

$$\boldsymbol{x} = \boldsymbol{x}^+ + \boldsymbol{x}^-$$

where $\boldsymbol{x}_0 \prec \boldsymbol{x}^+$, $\boldsymbol{x}^- \prec \boldsymbol{x}_0$, and

$$\boldsymbol{x}^+(E) = \begin{cases} \boldsymbol{x}(E \cap (0, \infty)) & \text{if} \quad 0 \notin E \\ \boldsymbol{x}(E \cup (-\infty, 0)) & \text{if} \quad 0 \in E \end{cases}$$

$$\boldsymbol{x}^-(E) = \begin{cases} \boldsymbol{x}(E \cap (-\infty, 0)) & \text{if} \quad 0 \notin E \\ \boldsymbol{x}(E \cup (0, \infty)) & \text{if} \quad 0 \in E \end{cases} \quad \text{for each} \quad E \in \mathscr{B}(\mathbb{R})$$

If $\boldsymbol{x}_0 \prec \boldsymbol{x}$, then $\boldsymbol{x}^+ = \boldsymbol{x}$ and $\boldsymbol{x}^- = \boldsymbol{x}_0$.

We propose to define the product of T_∞-fuzzy observables in the following way:

(1) For $x_0 \prec x$ and $x_0 \prec y$ we define the product $x * y$ by (17) in Definition 4.

(2) In other cases let us put

$$\begin{aligned} x * y &= (x^+ + x^-) * (y^+ + y^-) \\ &= [x^+ - (-x^-)] * [y^+ - (-y^-)] \\ &= x^+ * y^+ - (-x^-) * y^+ - x^+ * (-y^-) + (-x^-) * (-y^-) \end{aligned}$$

The observables $x^+, y^+, -x^-, -y^-$ are nonnegative and their products can be created by (17). The observables $-x^-$, $-y^-$ and the difference of observables are created by (4) and (5).

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